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OPTIMIZATION FORMULATION OF THE EVOLUTIONARY PROBLEM OF CRACK PROPAGATION UNDER QUASIBRITTLE FRACTURE

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For the evolutionary problem describing crack propagation in a solid with allowance for the irreversible work of plastic deformation due to the crack propagation, a general optimization formulation is proposed and investigated. For the optimum crack, data on the H^2-smoothnesses of the displacement field in the solid and, hence, on the finiteness of the stress at the crack tip, are obtained. The solvability of the optimization problem (i.e., the existence of an optimum crack) is proved for a curvilinear crack propagation path specified a priori. For the particular case of a straight path, a generalized criterion of crack growth is proposed. The question of the choice of a crack propagation path is discussed and a comparison with existing fracture criteria is made.

Key words: *crack, quasibrittle fracture, variational problem with a constraint, nonpenetration condition, optimization problem.*

Introduction. For an elastic solid body with a crack whose faces are free of stress, Leonov and Panasyuk [1] and Dugdale [2] proposed a crack model that, unlike the classical Griffith theory of brittle fracture, includes a plastic area modeled by coupling forces between the faces of this crack. Various physical mechanisms of interaction in the vicinity of the crack tip are postulated in [3, 4]. A detailed analysis of nonlinear fracture models is presented in [5–7].

A characteristic feature of the Leonov and Panasyuk and Dugdale crack model is the assumption that the crack faces are smoothly closed at its tip and the stresses are finite, unlike in the classical hypothesis, which postulates a $\sqrt{r}$-singularity for the displacement field and a $1/\sqrt{r}$-singularity for the stresses (r is the distance to the crack tip). The closure of the crack faces under plasticity conditions was also analyzed in [8].

Although the mechanical model of a quasibrittle fracture crack is widely used, up to now there is no accurate mathematical model that is described by a certain variational optimization problem and thus ensures the existence of cracks with specified properties for an arbitrary complex stress state. The present paper deals with constructing such a mathematical model and studying its properties.

In the variational problem considered, the total-energy function of a solid with a crack is represented in as the sum of the potential energy of the solid and the "surface energy" on the crack taking into account the irreversible work of plastic deformation. The distribution of the latter (the density function) depends on the crack opening by a characteristic elastoplastic diagram (see [9]). This leads to the requirement of nonnegativeness of the opening function (introduced previously in [10, 11]), which expresses the condition of mutual nonpenetration of the crack faces. The static variational problem of minimizing the total-energy function with a constraint was first formulated in [12]. The fundamental mathematical difficulty of this problem is that the minimized functional is nonconvex and nondifferentiable.

The novelty of the present work is the use of an optimization approach within the framework of a quasistatic formulation of the elasticity problem. It turns out that the static stress state found by solving the variational problem of minimizing the objective function of the total energy of the solid with an arbitrarily fixed crack is insufficient to

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validate the model. To ensure stress finiteness at the crack tip, it is necessary to additionally minimize the objective functional over all possible cracks. As a result, we obtain the evolutionary problem of optimizing the total-energy function with respect to the admissible crack displacements and shapes. The solution of this problem contains the crack growth (or closure) criterion and describes quasibrittle fracture.

The foundations of the optimization approach to describing the problem of Griffith crack propagation are formulated in [13]. The solvability of this problem for an anti-plane shear crack within the framework of continuous and time differentiable processes is analyzed in [14]. Nevertheless, the question of the solvability of the problem for an arbitrary crack topology and, hence, the question of the choice of a crack propagation path remain open. Time-discontinuous solutions of the problem of the quasistatic growth of an interface crack in a composite were obtained in [15] using numerical methods of path tracking.

1. Evolutionary Problem of Crack Propagation. We consider a bounded region $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$ which contains a crack Γ. We assume that Γ is a certain curve in $\mathbb{R}^2$ and that the case $\Gamma = \emptyset$ is possible. In the region of with the crack $\Omega \setminus \Gamma$, we consider the displacement vector $\boldsymbol{u} = (u_1, u_2)^{\mathrm{t}}(\boldsymbol{x})$ with $\boldsymbol{x} = (x_1, x_2)^{\mathrm{t}} \in \mathbb{R}^2$. The standard linear stress and strain tensors are defined by

$$\sigma_{ij}(\boldsymbol{u}) = c_{ijkl}\varepsilon_{kl}(\boldsymbol{u}), \qquad \varepsilon_{ij}(\boldsymbol{u}) = 0,5(u_{i,j} + u_{j,i}) \qquad (i, j = 1, 2)$$

with a positive definite symmetric tensor of the elastic coefficients c_{ijkl}, which can correspond to both homogeneous and inhomogeneous materials (the summation is performed over repeated indices $i, j, k, l = 1, 2$ and the index after the comma denotes differentiation with respect to the corresponding spatial coordinate). Assuming that the normal vector $\boldsymbol{\nu} = (\nu_1, \nu_2)^{\mathrm{t}}$ to the curve Γ is chosen, it is possible to distinguish the crack face Γ^+ and Γ^- so that its opening should be nonnegative:

$$[u_\nu] = u_i\nu_i\Big|_{\Gamma^+} - u_i\nu_i\Big|_{\Gamma^-} \geqslant 0 \qquad \text{on} \quad \Gamma. \tag{1}$$

The condition of mutual nonpenetration of the crack faces (1) is described in detail in [11].

For the specified load $\boldsymbol{f} = (f_1, f_2)^{\mathrm{t}}$ in Ω, we determine the total-energy function

$$E(\boldsymbol{f}, \boldsymbol{u}, \Gamma) = P(\boldsymbol{f}, \boldsymbol{u}, \Omega \setminus \Gamma) + S([u_\nu], \Gamma), \tag{2}$$

where

$$P(\boldsymbol{f}, \boldsymbol{u}, \Omega \setminus \Gamma) = \frac{1}{2} \int\limits_{\Omega\setminus\Gamma} (\sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u}) - f_i u_i)\, d\boldsymbol{x}; \tag{3}$$

$$S([u_\nu], \Gamma) = \int\limits_{\Gamma} \frac{2\gamma_0}{\delta_0} \min{(\delta_0, [u_\nu])}\, ds; \tag{4}$$

$\gamma_0 > 0$ and $\delta_0 > 0$ are specified material parameters. The functional P in (3) describes the potential energy of the solid with the crack, and the "surface energy" functional S in (4) depends on the crack opening defined in (1) and characterizes the irreversible work of plastic deformation. This model take into account a certain plastic zone Y on the crack Γ where $0 < [u_\nu] < \delta_0$ and the normal surface stresses $\sigma_\nu(\boldsymbol{u}) = \sigma_{ij}(\boldsymbol{u})\nu_j\nu_i$ reach the specified yield stress $2\gamma_0/\delta_0$.

Instead of the functional (4), the classical Griffith crack model uses the functional

$$S([u_\nu], \Gamma) = \int\limits_{\Gamma} 2\gamma_0\, ds, \tag{5}$$

which does not depend on the crack opening and characterizes its brittle fracture. Here the quantity γ_0 has the mechanical meaning of the specific surface energy of the two face Γ^+ and Γ^- of the crack Γ.

The crack model considered can be given the following mechanical interpretation. Let a slit Σ separates a compound solid body Ω into parts consisting of the same material. In the undeformed state, the slit faces are closely adjacent to one another. The friction between the adjacent surfaces is negligible. It is assumed that the slit faces Σ are drawn to one another by adhesion forces. The action of the load $\boldsymbol{f}$ determines the area $\Gamma \subseteq \Sigma$ on which opening of the slit occurs. In this case, Γ is separated into two nonintersecting sets: G (the opening $[u_\nu]$ is larger than the critical opening δ_0 when the adhesion forces disappear) and Y (the opening $[u_\nu]$ is smaller than the critical opening δ_0 when the adhesion forces draw the slit faces to one another). It is important to note that the formulation of the problem described here is urgent from the viewpoint of problems of modern nanomechanics.

We formulate the evolutionary problem for describing the crack propagation with time $t \geqslant 0$ as the optimization problem for each fixed $t > 0$: It is required to find $\Gamma(t) \in \Sigma(\Omega)$ that satisfies the relations

$$\begin{aligned}&\boldsymbol{u}(\Gamma(t)) \in H(\Omega \setminus \Gamma(t)) \text{ such that } [u_\nu(\Gamma(t))] \geqslant 0 \text{ on } \Gamma(t),\\ &E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma(t)), \Gamma(t)) \leqslant E(\boldsymbol{f}(t), \boldsymbol{v}, \Gamma(t)) \text{ for all } \boldsymbol{v} \in H(\Omega \setminus \Gamma(t))\\ &\text{such that } [v_\nu] \geqslant 0 \text{ on } \Gamma(t);\end{aligned} \tag{6}$$

$$\begin{aligned}&\Gamma(t) \supset \bigcup_{s<t} \Gamma(s),\\ &E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma(t)), \Gamma(t)) \leqslant E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma), \Gamma) \text{ for all } \Gamma \in \Sigma(\Omega)\\ &\text{such that } \Gamma \supset \bigcup_{s<t} \Gamma(s),\end{aligned} \tag{7}$$

where

$$\begin{aligned}&\boldsymbol{u}(\Gamma) \in H(\Omega \setminus \Gamma) \text{ such that } [u_\nu(\Gamma)] \geqslant 0 \text{ on } \Gamma,\\ &E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma), \Gamma) \leqslant E(\boldsymbol{f}(t), \boldsymbol{v}, \Gamma) \text{ for all } \boldsymbol{v} \in H(\Omega \setminus \Gamma)\\ &\text{such that } [v_\nu] \geqslant 0 \text{ on } \Gamma;\end{aligned} \tag{8}$$

$$\Gamma(0) = \Gamma_0. \tag{9}$$

Equality (9) is the initial condition at $t = 0$ with a specified initial crack $\Gamma_0 \in \Sigma(\Omega)$ (it is admissible that $\Gamma_0 = \varnothing$). Inequality (8) describes the true displacements for an arbitrary fixed crack $\Gamma \in \Sigma(\Omega)$. Inequalities (6) and (7) include the energetic criterion of crack growth (or closure).

The formulated optimization problem (6)–(9) contains great arbitrariness in the choice of admissible cracks $\Gamma \in \Sigma(\Omega)$ and, hence, it remains generally unsolved. The most general data on the existence of solutions of evolutionary problems of type (6)–(9) were obtained for a Griffith antiplane shear crack [14]. In Sec. 2, the issues related to the well-posedness of the problem (6)–(9) for a fixed crack Γ are investigated. In Sec. 3, the solvability of this problem [i.e., the existence of an optimum crack $\Gamma(t) \subset \Sigma$] is proved for the case of crack propagation along a curvilinear path $\Sigma \in \Sigma(\Omega)$ specified *a priori*.

2. Static Problem for a Fixed Crack. For a fixed crack $\Gamma \in \Sigma(\Omega)$, we consider the static problem of finding the true displacements $\boldsymbol{u}(\Gamma)$ among the admissible displacements $\boldsymbol{v} \in H(\Omega \setminus \Gamma)$ such that $[v_\nu] \geqslant 0$ on Γ. This problem is contained in formulations (6) and (8).

Formulation of the Problem and Its Well-Posedness. We assume that the crack Γ is specified in the form of a smooth curve. According to (1), we determine the set of admissible displacements

$$K(\Omega \setminus \Gamma) = \{\boldsymbol{v} \in H(\Omega \setminus \Gamma), \quad [v_\nu] \geqslant 0 \text{ on } \Gamma\},$$

where the space

$$H(\Omega \setminus \Gamma) \subset \{\boldsymbol{v} = (v_1, v_2)^{\mathrm{t}} \in H^1(\Omega \setminus \Gamma)^2\}$$

includes the condition $\boldsymbol{v} = \boldsymbol{0}$ of fastening of the solid body on the outer boundary $\partial\Omega$. For $\boldsymbol{f} \in L^2(\Omega)^2$, we consider the minimization problem: it is requited to find $\boldsymbol{u}(\Gamma) \in K(\Omega \setminus \Gamma)$ such that

$$E(\boldsymbol{f}, \boldsymbol{u}(\Gamma), \Gamma) \leqslant E(\boldsymbol{f}, \boldsymbol{v}, \Gamma) \quad \text{for all} \quad \boldsymbol{v} \in K(\Omega \setminus \Gamma) \tag{10}$$

[the function of the energy E is defined in (2) as the sum of P and S]. The first term is a positive definite quadratic (hence, convex and differentiable) function $\boldsymbol{u} \mapsto P(\boldsymbol{u})$, and the second term $S([u_\nu])$ is a nondifferentiable and nonconvex (concave) function. This is the reason for the difficulty of the analysis of (10).

Using the obvious nonnegativeness and Lipschitz continuity properties of the function $[u_\nu] \mapsto S([u_\nu])$ in (4) for $[u_\nu] \geqslant 0$, Kovtunenko [15] proved the following theorem.

Theorem 1. *There exists a solution $\boldsymbol{u}(\Gamma) \in K(\Omega \setminus \Gamma)$ of the nonconvex minimization problem with a constraint* (10).

It should be noted that: 1) the solution is not unique; 2) there are no optimality conditions (necessary and sufficient). Because of the absence of optimality conditions, we derive only the necessary conditions characterizing solution (10) in the form of a boundary-value problem.

Necessary Boundary Conditions. We introduce the following notation for the tangent vectors on Γ:

$$[\boldsymbol{u}_\tau] = [\boldsymbol{u}] - [u_\nu]\boldsymbol{\nu}, \qquad \sigma_\tau(\boldsymbol{u})_i = \sigma_{ij}(\boldsymbol{u})\nu_j - \sigma_\nu(\boldsymbol{u})\nu_i \qquad (i=1,2).$$

Theorem 2. *The solution of the problem* (10) *satisfies the relations*

$$-\sigma_{ij,j}(\boldsymbol{u}(\Gamma)) = f_i \quad (i=1,2) \quad \text{in} \quad \Omega, \tag{11}$$

$$\boldsymbol{u}(\Gamma) = \mathbf{0} \quad \text{on} \quad \partial\Omega \tag{12}$$

and the following relations on the crack Γ*:*

$$[\sigma_\nu(\boldsymbol{u}(\Gamma))] = 0, \qquad [\boldsymbol{\sigma}_\tau(\boldsymbol{u}(\Gamma))] = \mathbf{0}, \qquad \boldsymbol{\sigma}_\tau(\boldsymbol{u}(\Gamma)) = \mathbf{0}; \tag{13}$$

$$\begin{aligned} &\sigma_\nu(\boldsymbol{u}(\Gamma)) \leqslant 2\gamma_0/\delta_0 && \text{at} \quad [u_\nu(\Gamma)] = 0,\\ &\sigma_\nu(\boldsymbol{u}(\Gamma)) = 2\gamma_0/\delta_0 && \text{at} \quad 0 < [u_\nu(\Gamma)] < \delta_0,\\ &\sigma_\nu(\boldsymbol{u}(\Gamma)) = 0 && \text{at} \quad [u_\nu(\Gamma)] > \delta_0. \end{aligned} \tag{14}$$

Proof. The solution is assumed to be smooth enough. In (10), we take a trial function of the form $\boldsymbol{v} = \boldsymbol{u}(\Gamma) \pm \boldsymbol{\xi}$ with an arbitrary smooth function $\boldsymbol{\xi}$ such that $\boldsymbol{\xi} = \mathbf{0}$ on $\partial\Omega$ and $[\xi_\nu] = 0$ on Γ. Then, from (10) it follows that

$$\int\limits_{\Omega\setminus\Gamma} \Big(\sigma_{ij}(\boldsymbol{u}(\Gamma))\varepsilon_{ij}(\boldsymbol{\xi}) - f_i\xi_i\Big)\, d\boldsymbol{x} = 0.$$

Using Green's formula

$$\int\limits_{\Omega\setminus\Gamma} \sigma_{ij}(\boldsymbol{u}(\Gamma))\varepsilon_{ij}(\boldsymbol{\xi})\, d\boldsymbol{x} = -\int\limits_{\Omega\setminus\Gamma} \sigma_{ij,j}(\boldsymbol{u}(\Gamma))\xi_i\, d\boldsymbol{x} + \int\limits_{\partial\Omega} \sigma_{ij}(\boldsymbol{u}(\Gamma))n_j\xi_i\, ds$$

$$- \int\limits_{\Gamma} [\sigma_\nu(\boldsymbol{u}(\Gamma))\xi_\nu + \sigma_{\tau i}(\boldsymbol{u}(\Gamma))\xi_{\tau i}]\, ds \tag{15}$$

[$\boldsymbol{n} = (n_1, n_2)^{\mathrm{t}}$ is the outward normal to $\partial\Omega$], by virtue of the arbitrariness of the traces $\xi_\nu^+ = \xi_\nu^-$ and $\boldsymbol{\xi}_\tau^\pm$ on $\Gamma^\pm$, we obtain equalities (11) and (13).

According to the solution of the problem $\boldsymbol{u}(\Gamma)$, we separate the crack $\Gamma = Y(\boldsymbol{u}(\Gamma)) \cup G(\boldsymbol{u}(\Gamma))$ into two nonintersecting sets

$$Y(\boldsymbol{u}(\Gamma)) = \{\boldsymbol{x} \in \Gamma,\ 0 \leqslant [u_\nu(\Gamma)](\boldsymbol{x}) < \delta_0\}, \quad G(\boldsymbol{u}(\Gamma)) = \{\boldsymbol{x} \in \Gamma,\ [u_\nu(\Gamma)](\boldsymbol{x}) \geqslant \delta_0\}. \tag{16}$$

From the definition (16), it follows that $\boldsymbol{u}(\Gamma) \in K_{\boldsymbol{u}(\Gamma)}(\Omega \setminus \Gamma)$ and satisfies the inequality

$$\begin{gathered} E(\boldsymbol{f}, \boldsymbol{u}(\Gamma), \Gamma) \leqslant P(\boldsymbol{f}, \boldsymbol{v}, \Omega\setminus\Gamma) + S([v_\nu], Y(\boldsymbol{u}(\Gamma))) + S([v_\nu], G(\boldsymbol{u}(\Gamma)))\\ \text{for all} \quad \boldsymbol{v} \in K_{\boldsymbol{u}(\Gamma)}(\Omega\setminus\Gamma). \end{gathered} \tag{17}$$

Here

$$K_{\boldsymbol{u}(\Gamma)}(\Omega\setminus\Gamma) = \{\boldsymbol{v} \in K(\Omega\setminus\Gamma), \quad 0 \leqslant [v_\nu] \leqslant \delta_0 \text{ on } Y(\boldsymbol{u}(\Gamma)), \quad [v_\nu] \geqslant \delta_0 \text{ on } G(\boldsymbol{u}(\Gamma))\}.$$

Inequality (17) represents the minimization problem with a constraint for a convex differentiable functional that has a unique solution characterized by the following necessary and sufficient optimality condition for all $\boldsymbol{v} \in K_{\boldsymbol{u}(\Gamma)}(\Omega\setminus\Gamma)$:

$$\int\limits_{\Omega\setminus\Gamma} \Big(\sigma_{ij}(\boldsymbol{u}(\Gamma))\varepsilon_{ij}(\boldsymbol{v} - \boldsymbol{u}(\Gamma)) - f_i(v_i - u_i(\Gamma))\Big)\, d\boldsymbol{x} + \int\limits_{Y(\boldsymbol{u}(\Gamma))} \frac{2\gamma_0}{\delta_0}\, [v_\nu - u_\nu(\Gamma)]\, ds \geqslant 0. \tag{18}$$

Using Green's (15) formula and the relations (11)–(13) proved above, and integrating the volume integral (18) by parts, we obtain the following inequality for all $\boldsymbol{v} \in K_{\boldsymbol{u}(\Gamma)}(\Omega \setminus \Gamma)$:

$$\int\limits_{Y(\boldsymbol{u}(\Gamma))} \Big(\frac{2\gamma_0}{\delta_0} - \sigma_\nu(\boldsymbol{u}(\Gamma))\Big)[v_\nu - u_\nu(\Gamma)]\,ds - \int\limits_{G(\boldsymbol{u}(\Gamma))} \sigma_\nu(\boldsymbol{u}(\Gamma))[v_\nu - u_\nu(\Gamma)]\,ds \geqslant 0. \tag{19}$$

We fix a small number $0 < \varepsilon < \delta_0$. Let $[u_\nu(\Gamma)](\boldsymbol{x}) = 0$ at a certain point of the crack $\boldsymbol{x} \in \Gamma$. Setting $0 \leqslant [u_\nu(\Gamma)] \leqslant \varepsilon$ in a vicinity $O \subset \Gamma$ of the point $\boldsymbol{x}$, we choose a function χ on the crack Γ such that $\chi = 0$ on $\Gamma \setminus O$ and $0 \leqslant \chi \leqslant \delta_0 - \varepsilon$ on O. Then, substituting the expression $[v_\nu] = [u_\nu(\Gamma)] + \chi$ into (19), we obtain the inequality

$$\int\limits_O \Big(\frac{2\gamma_0}{\delta_0} - \sigma_\nu(\boldsymbol{u}(\Gamma))\Big)\chi\,ds \geqslant 0 \quad \text{for all} \quad \chi \geqslant 0 \tag{20}$$

and, hence, the first line in the boundary conditions on the crack (14).

We define the set

$$Y_\varepsilon = \{\boldsymbol{x} \in \Gamma,\ \varepsilon < [u_\nu(\Gamma)](\boldsymbol{x}) < \delta_0 - \varepsilon\}.$$

On the crack we choose a function χ such that $\chi = 0$ on $\Gamma \setminus Y_\varepsilon$ and $0 \leqslant \chi \leqslant 1$ on Y_ε. Then, $0 \leqslant [u_\nu(\Gamma)] \pm \varepsilon\chi \leqslant \delta_0$ on Y_ε. Substituting $[v_\nu] = [u_\nu(\Gamma)] \pm \varepsilon\chi$ as a trial function into (19), we obtain the equality

$$\int\limits_{Y_\varepsilon} \Big(\frac{2\gamma_0}{\delta_0} - \sigma_\nu(\boldsymbol{u}(\Gamma))\Big)\chi\,ds = 0 \quad \text{for all } \chi$$

and the second line in conditions (14).

Similarly, on the set

$$G_\varepsilon = \{\boldsymbol{x} \in \Gamma,\ [u_\nu(\Gamma)](\boldsymbol{x}) > \delta_0 + \varepsilon\}$$

we construct a patch function χ with the properties $\chi = 0$ on $\Gamma \setminus G_\varepsilon$ and $0 \leqslant \chi \leqslant 1$ on G_ε. Then, substituting the trial function $[v_\nu] = [u_\nu(\Gamma)] \pm \varepsilon\chi$ with $[u_\nu(\Gamma)] \pm \varepsilon\chi \geqslant \delta_0$ on G_ε into (19), we obtain the equality

$$-\int\limits_{G_\varepsilon} \sigma_\nu(\boldsymbol{u}(\Gamma))\chi\,ds = 0 \quad \text{for all } \chi$$

and the last line in (14).

Conditions (14) are used in the consideration of cracks according to the Leonov–Panasyuk and Dugdale models. We also note that conditions (11)–(14) are not sufficient for (10).

Additional Smoothness of the Solution on the Crack. We first obtain the necessary optimality conditions for the minimization problem (10). For this, we use the Lipschitz continuity property for the surface-energy functional

$$S([v_\nu], \Gamma) - S([u_\nu], \Gamma) \leqslant \int\limits_\Gamma \frac{2\gamma_0}{\delta_0}\Big|[v_\nu - u_\nu]\Big|\,ds \tag{21}$$

with arbitrary functions $\boldsymbol{u}$ and $\boldsymbol{v} \in K(\Omega \setminus \Gamma)$. Substituting $\boldsymbol{v} = (1-\alpha)\boldsymbol{u}(\Gamma) + \alpha\boldsymbol{\xi}$ as a trial function with an arbitrary $\boldsymbol{\xi} \in K(\Omega \setminus \Gamma)$ and the parameter $0 < \alpha < 1$ into inequality (10) divided by α and using estimate (21), by virtue of the differentiability of $\boldsymbol{u} \mapsto P(\boldsymbol{u})$, we have the following inequality for all $\boldsymbol{\xi} \in K(\Omega \setminus \Gamma)$:

$$\int\limits_{\Omega\setminus\Gamma} \Big(\sigma_{ij}(\boldsymbol{u}(\Gamma))\varepsilon_{ij}(\boldsymbol{\xi} - \boldsymbol{u}(\Gamma)) - f_i(\xi_i - u_i(\Gamma))\Big)\,d\boldsymbol{x} + \int\limits_\Gamma \frac{2\gamma_0}{\delta_0}\Big|[\xi_\nu - u_\nu(\Gamma)]\Big|\,ds \geqslant 0. \tag{22}$$

We use relation (22) to obtain the following result on the additional local smoothness of the solution $\boldsymbol{u}(\Gamma)$ outside the vicinity of the tips of the crack Γ.

Theorem 3. *For a smooth patch function ρ with $0 \leqslant \rho(\boldsymbol{x}) \leqslant 1$ and the support in the vicinity of $B(\boldsymbol{x}^0)$ of any strictly interior point $\boldsymbol{x}^0 \in \Gamma$ of the crack, the inclusion $\rho\boldsymbol{u}(\Gamma) \in H^2(\Omega \setminus \Gamma)^2$ is valid.*

Proof. For a smooth crack of the $C^{2,1}$-class there exists a local rectification function Γ in the vicinity of $B(\boldsymbol{x}^0)$ that can be represented as $x_2 = \varphi(x_1)$ $[x_1 \in I,\ \varphi \in C^{2,1}(I)]$. We define the discrete operators of tangential shear along the crack in $B(\boldsymbol{x}^0)$:

$$D_\tau^{\pm h} p = \frac{p^\varphi_{\pm h} - p}{hS^\varphi_{\pm h}}, \qquad S^\varphi_{\pm h}(x_1) = \sqrt{1 + \frac{(\varphi(x_1 \pm h) - \varphi(x_1))^2}{h^2}},$$

$$p_{\pm h}^{\varphi}(x_1, x_2) = p(x_1 \pm h, x_2 + \varphi(x_1 \pm h) - \varphi(x_1)) \qquad (h > 0).$$

For a sufficiently small parameter h, the function

$$\boldsymbol{\xi} = \boldsymbol{u}(\Gamma) - (h^2/2) S_{-h}^{\varphi} S_h^{\varphi} \rho D_\tau^{-h} D_\tau^h (\rho \boldsymbol{u}(\Gamma)) \tag{23}$$

belongs to the set $K(\Omega \setminus \Gamma)$ because, by virtue of $[u_\nu(\Gamma)](x_1, \varphi(x_1)) \geqslant 0$ and $[u_\nu(\Gamma)](x_1 \pm h, \varphi(x_1 \pm h)) \geqslant 0$, the following inequality holds on Γ:

$$[\xi_\nu] = (1 - \rho^2)[u_\nu(\Gamma)] + (1/2)\rho\Big(\rho_h^{\varphi}[u_\nu(\Gamma)]_h^{\varphi} + \rho_{-h}^{\varphi}[u_\nu(\Gamma)]_{-h}^{\varphi}\Big) \geqslant 0.$$

Substituting inequality (23) as a trial function into (22) divided by $h^2 S_{-h}^{\varphi} S_h^{\varphi}/2$, we obtain

$$\int\limits_{\Omega\setminus\Gamma} \sigma_{ij}\Big(D_\tau^h(\rho \boldsymbol{u}(\Gamma))\Big)\varepsilon_{ij}\Big(D_\tau^h(\rho \boldsymbol{u}(\Gamma))\Big)\, d\boldsymbol{x} \leqslant I_1 + I_2 + I_3, \tag{24}$$

where

$$I_1 = \int\limits_{\Omega\setminus\Gamma} \Big(\sigma_{ij}\Big(D_\tau^h(\rho \boldsymbol{u}(\Gamma))\Big)\varepsilon_{ij}\Big(D_\tau^h(\rho \boldsymbol{u}(\Gamma))\Big) - \sigma_{ij}(\boldsymbol{u}(\Gamma))\varepsilon_{ij}\Big(\rho D_\tau^{-h} D_\tau^h(\rho \boldsymbol{u}(\Gamma))\Big)\Big)\, d\boldsymbol{x},$$

$$I_2 = \int\limits_{\Omega\setminus\Gamma} \rho f_i D_\tau^{-h} D_\tau^h(\rho u_i(\Gamma))\, d\boldsymbol{x}, \qquad I_3 = \frac{2\gamma_0}{\delta_0}\int\limits_{\Gamma} \Big|\rho D_\tau^{-h} D_\tau^h(\rho[u_\nu(\Gamma)])\Big|\, ds.$$

Using the standard arguments of the shear operator [11], the integrals I_1, I_2, and I_3 can be estimated as follows:

$$I_1 \leqslant \text{const}\, \|\boldsymbol{u}(\Gamma)\|_{H^1(\Omega\setminus\Gamma)^2} \|D_\tau^h(\rho \boldsymbol{u}(\Gamma))\|_{H^1(\Omega\setminus\Gamma)^2},$$

$$I_2, I_3 \leqslant \text{const}\, \|D_\tau^h(\rho \boldsymbol{u}(\Gamma))\|_{H^1(\Omega\setminus\Gamma)^2}.$$

Then, Eq. (24) leads to the following estimate, which is uniform in h:

$$\|D_\tau^h(\rho \boldsymbol{u}(\Gamma))\|_{H^1(\Omega\setminus\Gamma)^2} \leqslant \text{const}.$$

This implies that $D_{\tau\tau}^2(\rho \boldsymbol{u}(\Gamma)), D_{\nu\tau}^2(\rho \boldsymbol{u}(\Gamma))$, and $D_{\tau\nu}^2(\rho \boldsymbol{u}(\Gamma)) \in L^2(\Omega \setminus \Gamma)^2$, where D_τ and D_ν are derivatives that are tangential and normal to the crack Γ. Accordingly,

$$D_\tau p = \frac{p_{,1} + \varphi' p_{,2}}{\sqrt{1 + (\varphi')^2}}, \qquad D_\nu p = \frac{p_{,2} - \varphi' p_{,1}}{\sqrt{1 + (\varphi')^2}}.$$

In $B(\boldsymbol{x}^0)$, Eq. (11) is locally representable as

$$D_{\nu\nu}^2 \boldsymbol{u}(\Gamma) = \boldsymbol{L}(D_{\tau\tau}^2 \boldsymbol{u}(\Gamma), D_{\nu\tau}^2 \boldsymbol{u}(\Gamma), D_{\tau\nu}^2 \boldsymbol{u}(\Gamma), D_\nu \boldsymbol{u}(\Gamma), D_\tau \boldsymbol{u}(\Gamma), f_\nu, f_\tau).$$

Then, $D_{\nu\nu}^2(\rho \boldsymbol{u}(\Gamma)) \in L^2(\Omega \setminus \Gamma)^2$, whence the statement of the theorem follows.

A corollary of Theorem 3 is the following lemma.

Lemma 1. *If in the vicinity of the crack tips* $[u_\nu(\Gamma)] = 0$ *and* $[\boldsymbol{u}_\tau(\Gamma)] = \boldsymbol{0}$, *then* $\boldsymbol{u}(\Gamma) \in H^2(\Omega \setminus \Gamma)^2$.

A proof of Lemma 1 is given in [11].

Smoothness of the Solution of the Evolutionary Problem. To obtain the smoothness of the solution in the vicinity of the crack tip, it is not sufficient to study the static problem (10) but it is necessary to consider the evolutionary optimization problem (6)–(9).

Theorem 4. *If a solution* $\Gamma(t)$ *of the problem* (6)–(9) *exists, then* $\boldsymbol{u}(\Gamma(t)) \in H^2(\Omega \setminus \Gamma(t))^2$.

Proof. We assume that there is a smooth continuation $\Gamma \in \Sigma(\Omega)$ of the crack $\Gamma(t)$ into the region Ω, i.e., $\Gamma(t) \subset \Gamma$. Because $K(\Omega \setminus \Gamma(t)) \subset K(\Omega \setminus \Gamma)$, it follows that

$$E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma), \Gamma) \leqslant E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma(t)), \Gamma(t)).$$

At the same time, the inverse inequality (7) holds, from which it follows that

$$E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma(t)), \Gamma(t)) = E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma), \Gamma). \tag{25}$$

Equality (25) implies that $\boldsymbol{u}(\Gamma(t)) \in K(\Omega \setminus \Gamma)$ is a solution of the stationary problem (10) with $\boldsymbol{f} = \boldsymbol{f}(t)$ for fixed t; $[u_\nu(\Gamma(t))] = 0$ and $[\boldsymbol{u}_\tau(\Gamma(t))] = \boldsymbol{0}$ on $\Gamma \setminus \Gamma(t)$. Therefore, Lemma 1 implies the statement of the theorem.

If the tip of the crack $\Gamma(t)$ is on $\partial\Omega$, the smoothness of the solution $\boldsymbol{u}(\Gamma(t))$ of the problem (6) at the tip follows from the corresponding boundary conditions specified on the outer boundary. Thus, Theorem 4 is proved.

3. Crack Propagation along a Specified Curvilinear Path. We consider the case of the optimization problem (6)–(9) where the crack path is specified *a priori* in the form of a certain smooth line $\Sigma \in \Sigma(\Omega)$. For example, if the data of the problem are symmetric about a certain straight line, it can be argued that the crack will propagate along this straight line.

Formulation of the One-Parameter Optimization Problem and Its Solvability. Let $0 \leqslant s \leqslant L$ be the arch length parameter along the curve Σ. We assume that one crack tip is fixed at $s = 0$ and that the position of the second tip $s = l$ defines the entire crack $\Gamma(l) \subset \Sigma$ as a function of the crack length parameter $0 \leqslant l \leqslant L$. In this case, the problem (6)–(9) becomes the one-parameter optimization problem: To find $l(t) \in [0, L]$ such that

$$\begin{aligned}&\boldsymbol{u}(l(t)) \in K(\Omega \setminus \Gamma(l(t))), \quad E(\boldsymbol{f}(t), \boldsymbol{u}(l(t)), \Gamma(l(t))) \leqslant E(\boldsymbol{f}(t), \boldsymbol{v}, \Gamma(l(t)))\\ &\text{for all} \quad \boldsymbol{v} \in K(\Omega \setminus \Gamma(l(t)));\end{aligned} \tag{26}$$

$$E(\boldsymbol{f}(t), \boldsymbol{u}(\Gamma(l(t))), \Gamma(l(t))) \leqslant E(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) \qquad \text{for all} \quad l \in [0, L], \tag{27}$$

where

$$\begin{aligned}&\boldsymbol{u}(l) \in K(\Omega \setminus \Gamma(l)), \quad E(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) \leqslant E(\boldsymbol{f}(t), \boldsymbol{v}, \Gamma(l))\\ &\text{for all} \quad \boldsymbol{v} \in K(\Omega \setminus \Gamma(l));\end{aligned} \tag{28}$$

$$l(0) = l_0 \tag{29}$$

for the specified initial crack Γ_0 of length $l_0 \in [0, L]$.

Lemma 2. *For fixed $t > 0$, the function of the reduced energy E as a function of the crack length l is semicontinuous from below, uniformly bounded, and monotonically decreasing:*

$$l \mapsto E(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) \in C([0, L)); \tag{30}$$

$$\begin{aligned}E(\boldsymbol{f}(t), \boldsymbol{u}(0), \Gamma(0)) &\geqslant E(\boldsymbol{f}(t), \boldsymbol{u}(l^1), \Gamma(l^1)) \geqslant E(\boldsymbol{f}(t), \boldsymbol{u}(l^2), \Gamma(l^2)\\ &\geqslant E(\boldsymbol{f}(t), \boldsymbol{u}(L), \Gamma(L)) \qquad \textit{for all} \quad 0 \leqslant l^1 \leqslant l^2 \leqslant L.\end{aligned} \tag{31}$$

Proof. Inequality (31) follows from the embeddings

$$K(\Omega \setminus \Gamma(0)) \subseteq K(\Omega \setminus \Gamma(l^1)) \subseteq K(\Omega \setminus \Gamma(l^2)) \subseteq K(\Omega \setminus \Gamma(L))$$

and inequality (28).

Let us prove the statement (30). We fix an arbitrary $0 \leqslant l < L$ and $s_0 > 0$ such that $l + s_0 \leqslant L$. According to Theorem 1, for any $0 \leqslant s \leqslant s_0$ there exists $\boldsymbol{u}(l+s) \in K(\Omega \setminus \Gamma(l+s))$ that satisfies the inequality

$$E(\boldsymbol{f}(t), \boldsymbol{u}(l+s), \Gamma(l+s)) \leqslant E(\boldsymbol{f}(t), \boldsymbol{v}, \Gamma(l+s)) \quad \text{for all} \quad \boldsymbol{v} \in K(\Omega \setminus \Gamma(l+s)). \tag{32}$$

Because $\boldsymbol{u}(l+s) \in K(\Omega \setminus \Gamma(l+s_0))$, substituting $\boldsymbol{v} = \boldsymbol{0}$ into (32) and using $S([u_\nu(l+s)], \Gamma(l+s)) \geqslant 0$ according to (4), we obtain the following estimate, which is uniform for all $0 \leqslant s \leqslant s_0$:

$$\|\boldsymbol{u}(l+s)\|_{H^1(\Omega\setminus\Gamma(l+s_0))^2} \leqslant \text{const.} \tag{33}$$

From the estimate (33), we derive the existence of a weak limit of the subsequence as $s \to 0$:

$$\boldsymbol{u}(l+s) \to \boldsymbol{u}^* \quad \text{weak in} \quad H(\Omega \setminus \Gamma(l+s_0)); \tag{34}$$

$$[u_\nu(l+s)] \to [u_\nu^*] \quad \text{strong in} \quad L^2(\Omega \setminus \Gamma(l+s_0)). \tag{35}$$

By virtue of $[u_\nu(l+s)] = 0$ on $\Gamma(l+s_0) \setminus \Gamma(l+s)$, from (35) it follows that $[u_\nu^*] = 0$ on $\Gamma(l+s_0) \setminus \Gamma(l)$ and $\boldsymbol{u}^* \in K(\Omega\setminus\Gamma(l))$. We take an arbitrary $\boldsymbol{v} \in K(\Omega\setminus\Gamma(l))$ as a trial function in (32) and pass to the lower limit as $s \to 0$, using the weak semicontinuity from below of the positive definite quadratic functional $P(\boldsymbol{f}(t), \boldsymbol{u}(l+s), \Omega \setminus \Gamma(l+s))$ from (3). By virtue of (34) and (35) for all $\boldsymbol{v} \in K(\Omega \setminus \Gamma(l))$, we obtain

$$\begin{aligned}E(\boldsymbol{f}(t), \boldsymbol{u}^*, \Gamma(l)) &\leqslant \liminf_{s\to 0} E(\boldsymbol{f}(t), \boldsymbol{u}(l+s), \Gamma(l+s))\\ &\leqslant E(\boldsymbol{f}(t), \boldsymbol{v}, \Gamma(l+s)) = E(\boldsymbol{f}(t), \boldsymbol{v}, \Gamma(l)).\end{aligned} \tag{36}$$

According to (36), $\boldsymbol{u}(l) = \boldsymbol{u}^*$ is a solution of the minimization problem (28). Inequality (31) implies the inequality

$$E(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) \geqslant \limsup_{s \to 0} E(\boldsymbol{f}(t), \boldsymbol{u}(l+s), \Gamma(l+s)). \tag{37}$$

From inequalities (37) and (36), it follows that

$$E(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) = \lim_{s \to 0} E(\boldsymbol{f}(t), \boldsymbol{u}(l+s), \Gamma(l+s)) \quad \text{for any } 0 \leqslant l < L. \tag{38}$$

Thus, Eq. (38) implies the proof of Lemma 2.

A corollary of Lemma 2 is the following theorem.

Theorem 5. *For any initial $l_0 \in [0, L]$ at each $t > 0$ there exists a solution $l(t) \in [0, L]$ of the optimization problem* (26)–(29).

By virtue of equality (25), Theorem 5 implies Lemma 3, which as an appendum to Lemma 2.

Lemma 3. *For the solution $l(t)$ of the optimization problem* (26)–(29), *the reduced-energy function satisfies the equality*

$$E(\boldsymbol{f}(t), \boldsymbol{u}(l(t)), \Gamma(l(t))) = E(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) \quad \textit{for all} \quad l(t) \leqslant l \leqslant L. \tag{39}$$

Case of a Straight Path. For the case of a straight ($\boldsymbol{\nu} = \text{const}$) crack path Σ, the differentiability of the reduced-energy functional with respect to the crack length parameter [12] is proved:

$$l \mapsto \frac{\partial E}{\partial l}(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) \in C(0, L). \tag{40}$$

The definition of the derivative in (40) is specified by the limit

$$\frac{\partial E}{\partial l}(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) = \lim_{s \to 0} \frac{E(\boldsymbol{f}(t), \boldsymbol{u}(l+s), \Gamma(l+s)) - E(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l))}{s}. \tag{41}$$

The representation of the derivative (41) is obtained in the form of two equivalent formulas. First, for an arbitrary (smooth) field of the kinematic velocity $\boldsymbol{V} = (V_1, V_2)^{\mathrm{t}}$ which is defined in Ω and $\boldsymbol{V} = \boldsymbol{0}$ on $\partial\Omega$ and tangential to the crack ($V_i \nu_i = 0$ on Σ), the following integral representation is valid:

$$\frac{\partial E}{\partial l}(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) = \int\limits_{\Omega \setminus \Gamma(l)} \Big(-\operatorname{div}(\boldsymbol{V} f_i) u_i(l) + \frac{1}{2} \operatorname{div}(\boldsymbol{V} c_{ijkl}) \varepsilon_{kl}(u(l)) \varepsilon_{ij}(u(l))$$

$$- \frac{1}{2} \sigma_{ij}(u(l))(u_{i,k}(l) V_{k,j} + u_{j,k}(l) V_{k,i}) \Big) \, d\boldsymbol{x} + \int\limits_{\Gamma(l)} \operatorname{div}(\boldsymbol{V}) \frac{2\gamma_0}{\delta_0} \min(\delta_0, [u_\nu(l)]) \, ds. \tag{42}$$

Second, integration of (42) by parts outside the vicinity B of the crack tip yields an equivalent representation that does not depend on the choice of B:

$$\frac{\partial E}{\partial l}(\boldsymbol{f}(t), \boldsymbol{u}(l), \Gamma(l)) = I_1(\boldsymbol{u}(l), \Omega \setminus B) - I(\boldsymbol{u}(l), \partial B) + \frac{2\gamma_0}{\delta_0} \min(\delta_0, [u_\nu(l)])|_{\partial B \cap \Gamma(l)}. \tag{43}$$

Here the first term

$$I_1(\boldsymbol{u}(l), \Omega \setminus B) = \int\limits_{\Omega \setminus B} \Big(\frac{1}{2} D_\tau(c_{ijkl}) \varepsilon_{kl}(u(l)) \varepsilon_{ij}(u(l)) - f_i D_\tau(u_i(l)) \Big) \, d\boldsymbol{x}$$

can be set equal to zero under the assumption that $c_{ijkl} = \text{const}$ and $\boldsymbol{f} = \boldsymbol{0}$ in B. The second term

$$I(\boldsymbol{u}(l), \partial B) = \int\limits_{\partial B} \sigma_{ij}(u(l)) \Big(\frac{1}{2} \varepsilon_{ij}(u(l))(q_k V_k) - D_\tau(u_i(l)) q_j \Big) \, ds \tag{44}$$

[$\boldsymbol{q} = (q_1, q_2)^{\mathrm{t}}$ is the outward normal to ∂B] is the Cherepanov–Rice integral, which is well-known in brittle fracture theory. The third term is defined at the point $\partial B \cap \Gamma(l)$ of intersection of the contour with the crack.

We note that in the case of a Griffith crack, according to (5), we have $I(\boldsymbol{u}(l), \partial B) = \text{const}$ irrespective of the integration path. In the case considered, Lemma 3 [equality (39)] and relation (43) imply the following theorem.

Theorem 6. *For a straight crack path Σ, the derivative* (41) *vanishes on the solution $l(t)$ of the optimization problem* (26)–(29)*:*

$$\frac{\partial E}{\partial l}(\boldsymbol{f}(t), \boldsymbol{u}(l(t)), \Gamma(l(t))) = 0; \tag{45}$$

by virtue of (45), *the equality*

$$I(\boldsymbol{u}(l(t)), \partial B) = \frac{2\gamma_0}{\delta_0} \min(\delta_0, [u_\nu(l(t))])\Big|_{\partial B \cap \Gamma(l(t))} \tag{46}$$

is valid; the integral I is defined in (44) *over an arbitrary smooth contour ∂B around the tip of a straight crack* $\Gamma(l(t))$.

From formula (46), it is possible to derive a generalized crack growth (or closure) criterion. Indeed, if $[u_\nu(l(t))] \geqslant \delta_0$ at a certain point $\partial B \cap \Gamma(l(t))$, then we have the equality $I(\boldsymbol{u}(l(t)), \partial B) = 2\gamma_0$, which coincides with the Cherepanov–Rice criterion for brittle fracture (where I is constant irrespective of the choice of the contour ∂B). Generally, from (46) it follows that I depends on ∂B, and it is possible to estimate $0 \leqslant I(\boldsymbol{u}(l(t)), \partial B) \leqslant 2\gamma_0$. Therefore, we determine the end point $\boldsymbol{x}_0$ of the plastic zone in the vicinity of the crack tip, where $[u_\nu(l(t))](\boldsymbol{x}_0) = \delta_0$, and formulate the fracture criterion in the form

$$I(\boldsymbol{u}(l(t)), \partial B) = 2\gamma_0$$

for an arbitrary contour ∂B such that $\partial B \cap \Gamma(l(t)) = \boldsymbol{x}_0$. It should be noted that the condition $[u_\nu(l(t))](\boldsymbol{x}_0) = \delta_0$ at a certain fixed point $\boldsymbol{x}_0$ of the crack is also used as a crack growth criterion [5].

On the Choice of the Crack Propagation Path. We fix a certain $t > 0$. Let $\Gamma(t)$ be a solution of the optimization problem (6)–(9). According to (13), the propagation path $\Sigma \in \Sigma(\Omega)$ of the crack $\Gamma(t) \subset \Sigma$ differs from its arbitrary smooth continuation into the region Ω by the condition of no tangential stresses

$$\boldsymbol{\sigma}_\tau(\boldsymbol{u}(\Gamma(t))) = \boldsymbol{0} \quad \text{on} \quad \Sigma. \tag{47}$$

This is a necessary condition for the choice of the crack propagation path Σ. In the particular case of the problem symmetric about a certain straight line Σ, the condition of the absence of tangential stresses is satisfied automatically for all $\Gamma \subset \Sigma$.

Let us illustrate the necessary conditions for the problem of nucleation of a curvilinear crack in a continuous solid Ω, i.e., $\Gamma_0 = \varnothing$. We choose a monotonically increasing load

$$\boldsymbol{f}(t) = t\boldsymbol{f}^0 \tag{48}$$

and assume that the solution of the problem (6)–(9) is

$$\Gamma(t) = \varnothing \quad \text{for all } 0 \leqslant t < t^*, \tag{49}$$

i.e., the solid remains continuous, without the occurrence of plastic zones, to a certain critical value $t = t^*$. We fix an arbitrary $t < t^*$. Then, from (6) it follows that $\boldsymbol{u}(\Gamma(t)) \in H_0^1(\Omega)^2$ and satisfies the inequality

$$E(t\boldsymbol{f}^0, \boldsymbol{u}(\Gamma(t)), \varnothing) \leqslant E(t\boldsymbol{f}^0, \boldsymbol{v}, \varnothing) \quad \text{for all} \quad \boldsymbol{v} \in K(\Omega \setminus \varnothing). \tag{50}$$

Obviously, the solution of the problem (50) is unique and, by virtue of (48), it can be represented in the form $\boldsymbol{u}(\Gamma(t)) = t\boldsymbol{u}^0$, which is linear in t and in which the function $\boldsymbol{u}^0 \in K(\Omega \setminus \varnothing) = H_0^1(\Omega)^2$ satisfies the inequality

$$E(\boldsymbol{f}^0, \boldsymbol{u}^0, \varnothing) \leqslant E(\boldsymbol{f}^0, \boldsymbol{v}, \varnothing) \quad \text{for all} \quad \boldsymbol{v} \in K(\Omega \setminus \varnothing). \tag{51}$$

For an arbitrary smooth crack $\varnothing \subset \Gamma \subset \Sigma$, from (25) it follows that $t\,\boldsymbol{u}^0 \in K(\Omega \setminus \Gamma)$ also satisfies (8). According to Theorem 2 applied to (51), and in view of (13), it is necessary that the following equality be satisfied:

$$\boldsymbol{\sigma}_\tau(\boldsymbol{u}^0) = \boldsymbol{0} \quad \text{on} \quad \Sigma; \tag{52}$$

in addition, from (20) with $O = \Sigma$, it necessarily follows that $t\sigma_\nu(\boldsymbol{u}^0) \leqslant 2\gamma_0/\delta_0$ on Σ, i.e.,

$$t^* \leqslant \frac{2\gamma_0}{\delta_0} \frac{1}{\max(0, \max_{\boldsymbol{x} \in \Sigma} \sigma_\nu(\boldsymbol{u}^0)(\boldsymbol{x}))}. \tag{53}$$

Therefore, the following theorem is valid.

Theorem 7. *Under monotonic loading* (48), *the necessary condition* (52) [*as a special case of* (47)] *characterizes the crack nucleation path Σ and inequality* (53) *is the upper-bound estimate of the critical time* $0 \leqslant t^* \leqslant \infty$ *before the occurrence of a plastic zone* (*or a cracks*) *in an initially continuous solid according to* (49).

Based on equality (47), it is also possible to seek the set $\Sigma(\Omega)$ of possible propagation paths Σ of a crack that is already available in the solid body Γ_0, after which condition (7) determines the true crack propagation path $\Gamma(t) \in \Sigma$, $\Gamma(0) = \Gamma_0$.

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